Solutions

to the puzzles in

Playing With Math:

Stories from Math Circles, Homeschoolers, and Passionate Teachers

Solutions by Sue VanHattum and Jack Webster

Please report any errors by emailing Sue at mathanthologyeditor@gmail.com.

Quick Links to each solution

Page 16, **Regions in a Circle (**from the Introduction)

Draw a circle, put two dots (or points) on it, and connect them with a straight line. You've split the circle into two regions. Add a third point, and connect all the points. Now there are four regions. How many regions will there be for n points?

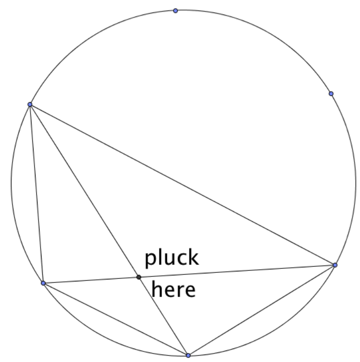
Hint from the book:

> Play with it first, by drawing circles with 3, 4, 5, and 6 points. Is the number of regions for 6 points surprising? It was to me. You might want to get data for 7 and 8 points. It won't have any obvious patterns, though. What makes new regions form? Thinking about that may move you toward a solution. If not, you can try figuring out how many regions get added by each new point. Those numbers are called first differences. The differences between those are second differences. When you find a pattern, you might try to build an equation, but it will be ugly. Go back now to that question about what makes new regions form.

Full solution:
(But remember, there are likely other ways to think about this.)

1. n points make $\frac{n(n-1)}{2}$ lines (each of the n points will have a line connecting it to each of the other $n-1$ points, but that would count each line twice, so divide by 2), and
2. Each line makes a new region each time it crosses another line and when it reaches its destination. (There's also the one original region before any lines are drawn.)
3. How do we count the line crossings? Some lines cross many others, and some cross none of the others. Hmm… There is an elegant way to see how to count these crossings, but I didn't see it right off, so I did a lot more work trying to use my data…
4. If you've figured the 1st and 2nd differences, and gone on looking for a pattern, to the 3rd and 4th differences, you noticed that the 4th differences were constant. A 4th difference is sort of like a 4th derivative, which is constant for a 4th degree polynomial, so perhaps a formula of the form $r = an^4 + bn^3 + cn^2 + dn + e$ will work (where n is the number of points used, and r is the number of regions formed).
5. If you plug in (1,1), (2,2), (3,4), (4,8), and (5,16) as values for *n and r*, you can find co-efficients (a, b, c, d, and e) for the equation by treating this as a system of 5 equations in 5 variables. I used a TI calculator to solve this. Changing the answers to fraction form gave me something ugly ($r = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1$), but then…
6. I know I want $r = 1 + \frac{n(n-1)}{2} + \textit{something}$, for the one original region, plus one region for each of the $\frac{n(n-1)}{2}$ lines, and then something more for each time two lines cross. (This is your last chance to try working more before peeking at my solution…)

7. I could write what I had as $r = 1 + \frac{n(n-1)}{2} + \frac{1}{24}\left(n^4 - 6n^3 + 11n^2 - 6n\right)$. It turns out that the part in parentheses is $n(n-1)(n-2)(n-3)$. Hmm…
8. Each crossing happens because of 4 points that make two lines that cross one time. (Re-read that line a few times, while imagining pulling the lines up as if they were strings in a game of cat's cradle.) *Every* collection of 4 points will produce 6 lines that cross exactly one time. How many collections of 4 points are there in *n* points? Exactly what we have: $\frac{n(n-1)(n-2)(n-3)}{24}$!

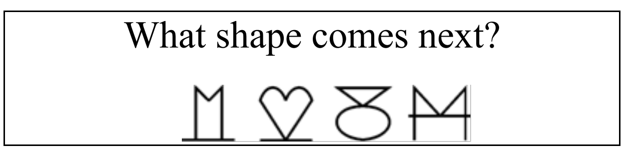


9. So the number of regions, r, is determined by the one original region plus a region for each time two lines cross plus an extra region when each line comes to its end. The prettiest way to write this (if you understand the notation) is that $r = 1 + {}_nC_4 + {}_nC_2$.

When I finally saw this for myself, I was awed by the beauty of it. If you weren't able to resist peeking at my answer, try the problem again in a few years. If you're at all like me, you will have forgotten enough that when you do solve it, you'll feel like you did it on your own.

This problem was on my mind for years, and I wrote a paper about my struggle with solving it. You can read that paper here: http://scholarship.claremont.edu/jhm/vol1/iss2/7/

Page 26, **Mirror Symmetry**

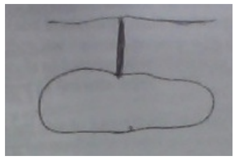


Hint from the book:

> From the footnote on page 26, "If you don't see what the next shape would be, cover the left half of each shape with your finger, and draw each right half on a piece of paper." If that's not enough, maybe it would help to know that what you just drew are four very common symbols.

Full solution:

What you see are the numbers 1, 2, 3, and 4, with their mirror images attached. So the next shape is a 5 with its mirror image attached on the left. Something like this:



Page 29, **Imbalance Abundance**

See diagrams in book.

Hints from the book:

> Number 2 has three inequalities, the left-hand side, the right-hand side, and the Here are some hints for the puzzles I found hardest: Number 2 has three inequalities: the left-hand side, the right-hand side, and the whole thing. The tilt at the top tells us that the left-hand side weighs more. In number 4, I got stuck at something that now seems obvious. If the square is heavier than two circles (let's write that as s>2c), then it's certainly heavier than one circle. (Using algebraic notation like s>2c may help on some of the problems.) On number 12, I needed a hint from the author: "Looking at the lone circle, it hangs down, so circle must have positive weight."

Full solution:
2. From the left-hand side, we have square > circle (> means heavier). From the right-hand side, we have triangle > circle, from the upper balance point, we have square > triangle. So square > triangle > circle, or equivalently, circle < triangle < square.

3. square < triangle < circle
4. circle < square < triangle
5. circle < square < triangle
6. circle < square < triangle
7. circle < triangle < square
8. circle < triangle < square
9. triangle < square < circle
10. square < circle < triangle
11. triangle < square < circle

12. What! On the right-hand side, how can one circle weigh more than two circles plus something else?! When we weigh objects, this doesn't happen. But if the shapes represent any numbers at all, then something must be negative. Is it the circle or the square? We get triangle > circle from the upper balance point. (s+2t>c and c>2c+s gives s+2t>s+2c, which gives 2t>2c, or t>c.) Looking at the lone circle, it hangs down, so circle must have positive weight. That means square must be the negative one. Now we have square < circle < triangle.

Page 38, **Crossing the River #1**

A farmer is traveling with a chicken, grain, and a fox. She has to cross a river and can only take one across at a time. If she leaves the fox and chicken together, the fox will eat the chicken. If she leaves the chicken and grain together, the chicken will eat the grain. Can she get them all to the other side?

Hint from the book:

What can stay alone? Move this as much as you need to.

Full solution #1:
G = Grain,
C = Chicken,
F = Fox
:) = Farmer

Observe only F&G can be together unsupervised by the farmer. The solution must be to force that to happen.

SIDE A	SIDE B
CGF :)	
GF	C :)
GF :)	C
G	FC :)
CG :)	F
C	GF :)
C :)	GF
	CGF :)

Full solution #2:
The fox and grain can stay together. The chicken cannot stay with either the fox or the grain. If the farmer is going to get in that boat, the only thing she can do is to take the chicken with her. She leaves the chicken on the other side and comes back. Now she can take either the grain or the fox, it doesn't matter. She brings it across. Here's the least obvious step. She leaves it, and brings the chicken back. She leaves the chicken on the first side, and brings the other (of fox and grain) across. She comes back to get that chicken, and brings it across again. Done.

Page, 39, **Crossing the River #2**

A group of five adult dinosaurs and two young dinosaurs need to cross a river. A small boat that can hold either one adult or two youngsters is available. Everyone can row the boat. Can the whole group get across? If so, how many trips are necessary to get everyone across the river? (Count a trip as just going from one side to the other.)

Harder:

1. *What if there were still two young dinosaurs, but they were with a hundred adults? [Hint: Try groups that have two youngsters and one, two, three or more adults. Can you see a pattern?]*
2. *What if there were still two young dinosaurs, but they were with* ***n*** *adults?*

Play with it:
What if there were more youngsters?
Can you modify the problem in other ways?

Hint from the book:

> Since taking an adult or one child across first would do no good (they'd just have to come back), you know the first crossing must be the two kids. Then one kid can bring the boat back. What can you do next? Each step of the way, try things, and see if they get you to a new situation. Once you've gotten one adult across, can you repeat what you did?

Full solution:
We solve the general problem, as the other parts follow from this.

Suppose there are 2 children and N adults.
Denote the boat by B. Then:

Side A				Side B	
Adults	Children			Adults	Children
N	2	B		0	0
N	0		B	0	2
N	1	B		0	1
N-1	1		B	1	1
N-1	2	B		1	0

And this state is exactly as before, except with N-1 instead of N. The five rows of this table represent the starting position and four crossings of the boat. If we made four more crossings, we'd have N-2 adults on the left and 2 adults on the right.

When N = 0, we can complete the trips with the following one crossing:

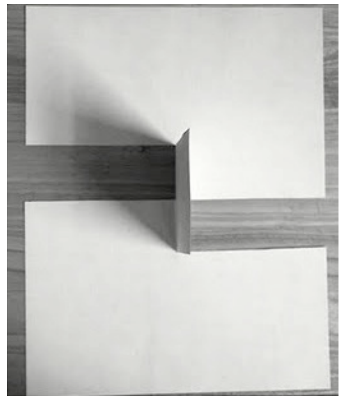
0	0		B	0	2

And so there are 4*N + 1 crossings required *at most*. It is easy to convince yourself that in fact it cannot be done with fewer crossings. So with 5 adults, we'd need 21 crossings, and with 100 adults we'd need 401 crossings.

Adding more children just adds to the final stage.

Page 44, **Foxes and Rabbits**

In this meadow some foxes and rabbits are hiding.
Each has very good eyes, and can see all the others.
Each fox can see twice as many rabbits as foxes. (Yum!)
Each rabbit can see an equal number of foxes and rabbits. (Yikes!)
How many are in the meadow?



Hint from the book:

> Since the numbers are small, drawing a picture might help. Show the fox who's looking. And show what the fox sees. One of the rabbits seen is also looking, what does the rabbit see?

Full solution:
If the first fox sees one fox and two rabbits, then one of the rabbits will see one other rabbit and two foxes. That's not it. What would you try next?

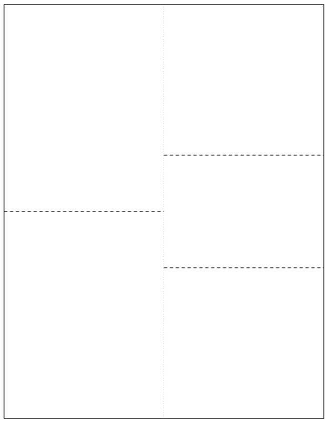
If the first fox sees two foxes and four rabbits, one of the rabbits will see three other rabbits and three foxes. That's it!

Page 49, **Is This for Real?**

Hint from the book:

> Imagine picking this up and pushing the front half down on the left (straightening the cut that goes partway through the middle of the paper) and the back half down on the right (again straightening the cut that goes partway through the middle of the paper, so those two edges will be aligned). Now figure out where the cuts are, and cut your paper the same way.

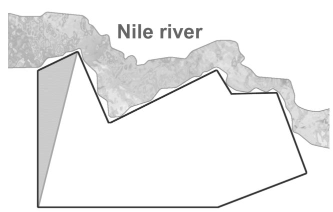
Full solution #1:
If you still couldn't see it, here's a diagram of the paper. Draw a pencil line along the middle (long way). Cut to that center line along the middle (short way) from the left, and cut to that center line a bit above and below the middle from the right. Fold the middle tab up. Now take the far end of the paper (top in this diagram), and turn it over.



Full solution #2:
Fold lightly along the long side. On one side of the fold, cut at the middle (of the long side) to the fold line. On the other side, cut about 1 ½ inches above the middle, to the fold line, and again 1 ½ inches below the middle. You've created a flap. Hold it up, and fold so the rest of the paper folds to the left at the bottom and to the right at the top.

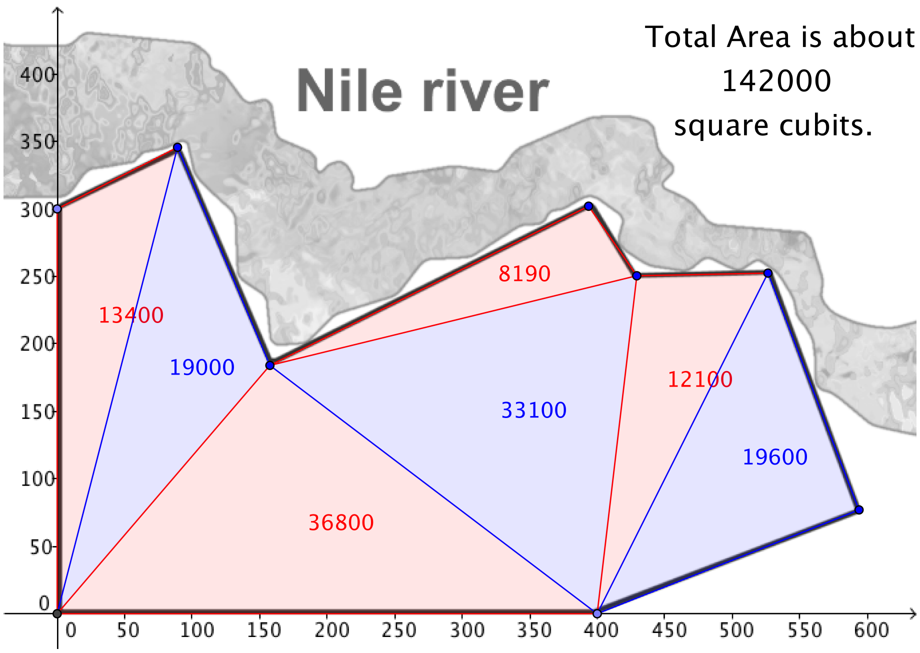
Page 60, **Egyptian Farm Puzzle**

Hint from the book:



> Copy the drawing, and make a paper measuring device with 100 units equal to the length of the upper left line of the farmer's property. Now draw six lines inside, so there are seven triangles. One way to start is shown to the right. (There are many ways to triangulate, which will all give the same final area.) Now, for each triangle, measure the length of one side (base), and from that use a perpendicular line to the opposite vertex to measure the height. (Or do it the way I did, and use Geogebra to draw in lines and calculate areas.)

Full solution:



Page 72, **Vertices, Edges, and Faces**

Do you see any patterns in your data?
How many Platonic solids can you find?

Hint from the book:

> There is a simple equation relating the number of vertices, edges, and faces. Try various combinations of adding and subtracting until one stays constant.
> There are 5 Platonic solids. Can you find a reason why there cannot be any more? How many faces for each one?

Full solution:
It turns out that if you count the vertices (corners) and call this number V, count the edges and call this E, and count the faces and call this F, you get V-E+F=2, for any shape. For example, the cube has 8 vertices, 12 edges, and 6 faces, and 8-12+6 = 2. This is called Euler's formula, because Leonard Euler proved that it is always true.

There are five Platonic solids:

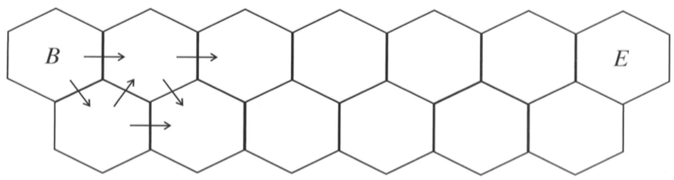
- the tetrahedron is a pyramid with a triangular base, giving it four triangular faces,
- the cube has six square faces,
- the octahedron has eight triangular faces,
- the dodecahedron has twelve pentagonal (5 edges) faces, and

- the icosahedron has twenty triangular faces.

If six triangles meet at a vertex, they will lie flat. So we can use three, four, or five triangles at a vertex - giving the tetrahedron, octahedron, and icosahedron – but not six. Four squares meeting at a vertex lie flat, so our only choice with squares is to have three meet at a vertex (cube). Three pentagons can also meet at a vertex (dodecahedron), but three hexagons meeting at a vertex would lie flat, so no other shapes work.

Page 76, **Honeybee (#1)**

A honeybee living on the hexagonal comb pictured wishes to move, but is constrained to always have an eastward (left to right) component to her motion, as shown by the arrows. Given this constraint how many different paths are there from cell B to cell E?



Hint from the book:

> Put a number in each cell that describes how many ways to get that far. There is only one way to get to the first cell in the bottom row, so put a 1 there. The first cell after the B in the top row can be arrived at two ways – directly from B, and also from the bottom row, so put a 2 there. How many ways are there to get to the next two cells?

Full solution:
The second cell on the bottom can be arrived at from the first bottom cell (1 way) or from the top cell (2 ways), so there are 3 ways to get there. The next top cell can be arrived at from the top (2 ways) or the bottom (3 ways). Each time we can just add the numbers of the two cells that can bring us to this cell. We get 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233. There are 233 ways to arrive at the E cell.

Page 76-77, **Ones and Twos (#2)**

*How many ways are there to write the number 3 as a sum of 1's and 2's if order matters (that is, 1+2 is different from 2+1) and you have two different kinds of 1s (say, thick and thin)? By way of example, there are 5 such ways to write the number 2: 1+1, 1+**1**, **1**+1, **1**+**1**, 2.*

Hint from the book:

First figure our how many ways to get 2.

Full solution:
There are two ways to produce a one, thick or thin. There are five ways to get 2 (1+1, 1+**1**, **1**+1, **1**+**1**, 2). To get 3 we can:

- do 2 plus 1 (one of two ways),
- do 1 (one of two ways) plus 2,
- do 1 plus 1 plus 1, eight ways (thick or thin for each one).

So there are 12 ways total.

Alternate description:
Let us first solve it with just a single 1. Then we would have:
1+1+1,
1+2,
2+1.

Of course, for each occurrence of one there are actually 2 possible representations, so there are:

Pattern	Number
1+1+1	8
1+2,	2
2+1,	2
Total	12

Page 77, **Abeeba (#3)**

There is a language called ABEEBA which has only three letters, A, B, and E. Words are formed in the usual way, by concatenating letters, with the exception that "AE" is not allowed to appear. So, for example, AA, AB, BA, BB, BE, EA, EB, EE, is a complete list of the two letter words of ABEEBA. How many five-letter words are there?

Hint from the book:

First figure out how many two-letter words, and how many of those end in A. From there, figure out how many three-letter words, and how many of those end in A. You may begin to see a pattern.

Full solution:
Abeeba has 8 two-letter words, three which end in A, five which do not.
Three-letter words:

- 2*3 for the two letters that can follow the three two-letter words ending in A, plus
- 3*5 for the three letters that can follow the five two-letter words not ending in A,

- giving 6+15 = 21 ways to make a three-letter word (8 of which end in A).

Four-letter words: 2*8+3*13=55 (21 of which end in A)
Five-letter words: 2*21+3*34= 144 five-letter words.

Page 78, **Math Without Words #1**

Hint from the book:

> Color in the pieces that you're trying to imagine laid out straight, as if they were pieces of spaghetti or ring noodles.

Full solution:
The fourth puzzle simplifies to two lines.
#5 is two loops.
#6 is one line.
#7 is one line.
#8 is two loops.
#9 is a loop and a line.
#10 is three loops.

Page 84, **Math Without Words #2**

Hint from the book:

> Can you see a way to organize the 8 different layouts, so they somehow go with what's come before? If you want to cover 5 squares with blocks of width one and two, then the last block is either one or two. If it's one, what comes before it covers four blocks. If it's two, what comes before it covers three blocks.

Full solution:

Each number on the left is represented on the right partitioned into blocks of size 1 and 2, where order matters.

So 2 = 1+1 as well as just 2, which gives a total of 2 ways to partition it.
3 = 1+1+1, 1+2, 2+1 (3 ways)
4 = 1+1+2, 1+2+1, 2+1+1, 2+2, 1+1+1+1 (5 ways)

5 can be represented as:

1+1+1+1+1	2+1+1+1
1+1+1+2	2+2+1
1+1+2+1	2+1+2
1+2+1+1	1+2+2

which gives a total of 8 ways, as the answer indicates.

Now, for 6, we aren't given a total. How can we make sure we find all the ways? If you've played with lots of math puzzles you may recognize the sequence that the answers make (2, 3, 5, 8). If not, you might want to think about how to use your previous lists to make the new partitions. You can add a 1 at the end of each partition in the list for 5, to get 6. What about partitions that end with a 2? Use the list for 4 to get those. Each one is guaranteed to be different, and if we found everything for 4 and 5, then we have now found everything for 6. How many ways? 5+8, or 13.

For 7, we would have 21 ways to partition into blocks of size 1 or 2.

Page 91, **Food for Thought**

Anabel: "I'm thinking of four numbers."
Bessy: "I'm thinking of when we're going to eat."
Anabel: "The second of them is a positive integer."
Bessy: "If there are five cows staring at you, are two cows staring at you?"
Anabel: "The sum of the first two equals the third. The sum of the second and the third equals the fourth."
Bessy: "Hmm… What is the sum of the third and the fourth?"
Anabel: "The first number."
Bessy: "Bummer!"
Anabel: "No, 'number'."
Bessy: "What are the four numbers?"
Anabel: "Who cares? What I would like to know is how big can their sum be?"

Hint from the book:

> Algebra will be our superhero here. We're tying to figure out the sum of four numbers. Let's call the numbers A, B, C and D. What have the cows said about the numbers? Can we write that with our variables? (Warning: The phrase 'how big' is sneaky here.)

Full solution:

- Call the four numbers A, B, C, and D.
- Make equations from what Anabel and Bessy say about them: A+B=C, B+C=D, C+D=A
- (There's also an inequality: B>0)
- Add together the three equations: (A+B) + (B+C) + (C+D) = C+D+A, or A+2B+2C+D = A+C+D, or 2B + C = 0, giving us C = -2B.
- By the first equation, A+B = -2B, so A = -3B.
- By the second equation, B + -2B = D, so D = -B.
- We can now notice that the third equation is consistent (C+D = -2B + -B = -3B = A), which is good.
- Since B is greater than 0, all the other numbers are less than 0.
- The sum of all four numbers is -3B + B + -2B + -B = -5B, which is less than zero.

- The phrase ‘how big’ is a bit sneaky here. The biggest negative numbers are those closest to zero (like the warmest temperatures below zero are those closest to zero).
- So the biggest the sum can be is when B is the smallest possible positive integer, 1, which gives a sum of -5.

Page 95, **St. Mary’s Math Contest Sampler**

1. A company named JULIA has an advertising display with just the five letters of its name, lit up in various colors. On a certain day the colors might be red, green, green, blue, red. The company wishes to have a different color scheme for each of the 365 days of the year. What is the minimum number of colors that can be used for this purpose?
2. How would you decide whether a number in base 7 is even, based on its digits?
3. Given the sequence 1, 2, 4, 5, 7, 8, 10, ... where every third integer is missing, find the sum of the first hundred terms in the sequence.
4. Find the sum of the cubes of the numbers from 1 to 13. Now find the sum of the cubes of the numbers from 1 to n.
5. Using exactly five 5's, and the operations +, −, ×, ÷, and factorial (!), represent each of the numbers up to 30.

Hints from the book:

1. Start with just two colors, and begin a list. What do you notice? I notice that each letter has two possibilities.
2. What will the value of an odd digit in the sevens place be? Make a few examples for yourself, and look for patterns.
3. How do we find the 100^{th} term in the sequence? Do you know how to add all the numbers from 1 to that number? Can you add pairs of numbers together to get a sum that’s always the same? Once you have that sum, you can subtract the sum of 3+6+…+last left out number.
4. Make a table: first column, numbers 1 to 13; second column, their cubes; third column, the running sum. You may notice a pattern in this column.
5. This one just takes lots of play, experimentation, and persistence.

Full solution:
1. With two colors, each of the five letters can be one of two colors, and there will be 2^5 possibilities. That’s 32. If we have three colors, each letter can be one of the three, giving 3^5 or 243 possibilities. Not quite enough. But 4^5 is 1024, which is more than enough. So we need four colors.

2. Since 7 to any power is odd, each odd digit represents an odd number, and (odd times even being even) each even digit represents an even number. So the sum of the digits must be even, or equivalently, there must be an even number of odd digits.

3. This sequence uses 2/3rds of the integers; 2/3rds of 150 is 100. So our 100^{th} number will be 149. To add up all of the first 149 numbers, imagine writing the sum out twice,

with the second sum going from 149 down to 1 and written below the first. Adding down, we have 149+1, 148+2, etc, with each little sum being 150. There are 149 little sums, so we get 22350. But we added twice what we wanted, so our answer is 11175. This can be generalized: The sum of the numbers from 1 to *n* is $\frac{1}{2}n(n+1)$. Now we have to subtract out the multiples of 3: 3+6+9+ … +147 = 3(1+2+3+ … + 49) = 3/2*50*49 = 3675.
So we get 11175-3675 = 7500.

4. The sum of the cubes of 1 through 13 is 8281. To get the sum of the cubes of 1 through *n*, we need to find a pattern. We can make a table with the first column being the numbers from 1 to *n*, the next column being their cubes, and the next being the sum so far. At this point, you might notice that all the numbers you're getting are perfect squares. Let's try taking the square roots in a new column. The numbers in this last column might be familiar. (If you had used this technique to see the pattern when the numbers from 1 to n are added, as in problem 1, you would have seen this last column as your sum so far: 1+2 = 3, 1+2+3 = 6, 1+2+3+4 = 10, etc.) So the last column is $\frac{1}{2}n(n+1)$, making the previous column $\frac{1}{4}n^2(n+1)^2$. This is the answer, but proving that it will always work is harder, and seeing why it works is another matter altogether. This page may be helpful if you'd like to pursue that: jaxwebster.wordpress.com/2009/12/23/sum-of-cubes

5. There are many possibilities for most numbers. This is one possible set of answers:

$$1 = \frac{5+5}{5} - \frac{5}{5}$$

$$2 = \frac{5+5+5-5}{5}$$

$$3 = \frac{5+5}{5} + \frac{5}{5}$$

$$4 = \frac{5+5+5+5}{5}$$

$$5 = 5+5+5-5-5$$

$$6 = 5 + \frac{5}{5} + 5 - 5$$

$$7 = 5 + \frac{5}{5} + \frac{5}{5}$$

$$8 = 5 + \frac{5+5+5}{5}$$

$$9 = \frac{5!}{5} - 5 - 5 - 5$$

$$10 = 5\left(\frac{5}{5} + \frac{5}{5}\right)$$

$$11 = \frac{5!}{5+5} - \frac{5}{5}$$

$$12 = 5 + 5 + \frac{5+5}{5}$$

$$13 = \frac{5!-5}{5} - 5 - 5$$

$$14 = 5 + 5 + 5 - \frac{5}{5}$$

$$15 = 5+5+5+5-5$$

$$16 = 5 + 5 + 5 + \frac{5}{5}$$

$$17 = \frac{5!-5-5}{5} - 5$$

$$18 = \frac{5!-5\cdot 5-5}{5}$$

$$19 = 5\cdot 5 - 5 - \frac{5}{5}$$

$$20 = 5\cdot 5 - 5 + 5 - 5$$

$$21 = \frac{5!+5+5}{5} - 5$$

$$22 = \frac{5!}{5} - \frac{5+5}{5}$$

$$23 = \frac{5!-5}{5} + 5 - 5$$

$$24 = \frac{5!+5}{5} - \frac{5}{5}$$

$$25 = \frac{5!+5}{5} + 5 - 5$$

$$26 = \frac{5!}{5} + \frac{5+5}{5}$$

$$27 = \frac{5!-5-5}{5} + 5$$

$$28 = \frac{5!}{5} + 5 - \frac{5}{5}$$

$$29 = \frac{5!}{5} + 5 + 5 - 5$$

$$30 = \frac{5!}{5} + 5 + \frac{5}{5}$$

---

page 95, **The Candy Conundrum**

1. *You have 5 red apple candies. How many different nonempty sets of candies could you put in your mouth?*
2. *You have 5 red apple and 4 green lime candies. How many different nonempty sets of candies could you put in your mouth?*
3. *You have 5 red apple, 4 green lime, and 3 yellow pineapple candies. How many different nonempty sets of candies could you put in your mouth?*
4. *You have 5 red candies. How many flavors could you make?*
5. *You have 5 red and 4 green candies. How many flavors could you make?*
6. *You have 5 red, 4 green, and 3 yellow candies. How many flavors could you make?*
7. *Expressed geometrically, what does it mean for two different sets of candies to be the same flavor? Assume for now that there are only two colors.*
8. *Describe a geometric way of understanding how many different flavors there are. Compare it to the numeric approach.*
9. *What does symmetry tell you about the number of flavors with k red candies and k green candies?*
10. *If you have 1 candy of each of n colors, how many different flavors are possible? If you have k candies of 1 color, how many different flavors are possible? OK, sorry, that was too easy.*

11. *If you have 2 candies of each of n colors, how many different flavors are possible? If you have k candies of each of 2 colors, how many different flavors are possible?*
12. *Generalize as much as you can!*
13. *How do the previous answers change if the candies are large, so there is an upper limit to how many can fit in your mouth at once?*
14. *What can you say about the relative probability of various flavors if you pick a random handful of size n out of a set of candies? Start by considering some easy cases, where n is small, and there aren't too many different flavors, and plenty of candies of each flavor (since n will limit you, it gets more complicated if you also have limits due to running out of candies).*

Hints from the book:

2. You can have anywhere from 0 to 5 red, along with anywhere from zero to 4 green, except we don't count zero of both.

Warning: The higher numbered problems are meant to be hard. The organizers of the Julia Robinson Mathematics Festival wanted some questions in each set to be easy enough for everyone, and a few to be hard enough to be worthy of mathematical research.

Full solution:
1. 5
2. (5+1)(4+1) - 1 = 29
3. (5+1)(4+1)(3+1) - 1 = 119
4. 1
5. 17

Let (x,y) represent (number of reds, number of greens). So $0 \le x \le 5, 0 \le y \le 4$.

We have:

- (0,1) which tastes like lime alone
- (1,0), (1,1), (1,2), (1,3), (1,4) the first tastes like apple alone, and the others get more limey as you add greens
- (2,1), (2,3) because (2,0) tastes just like (1,0), (2,2) tastes just like (1,1), and (2,4) tastes just like (1,2)
- (3,1), (3,2), (3,4)
- (4,1), (4,3)
- (5,1), (5,2), (5,3), (5,4)

We get 1+5+2+3+2+4 = 17.

6. 94

Let (x,y,z) represent (number of reds, number of greens, number of yellows). So $0 \le x \le 5, 0 \le y \le 4, 0 \le z \le 3$.

We have:

- (1,0,0), (0,1,0), and (0,0,1)
- (x,y,0) where x,y are relatively prime, non-zero; 15 of these - from the previous problem, not counting (0,1) and (1,0)
- (x,0,z), as above, but z can't be 4, so take out the three entries with y=4 from above (12 of these)

- (0,y,z) as above, but y can't be 5, so take out three more entries (9 of these)
- any of the fifteen non-zero entries from the previous problem, with 1,2, or 3 as the third entry (15*3 of these)
- (2,2,1), (2,2,3), (3,3,1), (3,3,2), (4,4,1), (4,4,3)
- (2,4,1), (2,4,3)
- (4,2,1), (4,2,3)

We get 3+15+12+9+15*3+6+2+2 = 94.

7. Associate the combination made of *x* red and *y* green sweets with the point (x,y). Now identical flavors make the same angle with the axis. Or you could say they are on the same line through the origin.

8. Draw the lattice points corresponding to each number pair. For each point, draw a ray from the origin though that point, and remove all other points on that ray.

9. (k,k) is the same flavor as (1,1), so if there are no other colors, there is only one flavor like this.

10. 1 sweet of each of n colors gives 2^n-1 flavors - as each flavor can be present or absent, but the 0 flavor doesn't count.

Obviously if you have k of one color, you just have one flavor.

11. n colors where you have 2 of each: There are 3^n combinations. But we ignore all those which involve only 0's and 2's, as then you can divide by 2 to get 0's and 1's. Hence we end up with 3^n - 2^n. This also takes care of the 0 case.

2 colors where you have k of each:
This answer is from Jack Webster. (Sue VanHattum, editor of the book, does not understand it.)
You have (1,i) and (i,1) for all 0 <= i <= k except i = 1 (which are counted in the next term) and (a,b) where a and b are coprime.
and (1,1)

So answer = 2k + 1 + Sum[EulerTotient(i) , {i,2,k}]

so for instance, when k = 3 there is:
2*3 + EulerTotient[2] + EulerTotient[3] = 6 + 1 + 2 = 9

when k = 4:
2*4 + EulerTotient[2] + EulerTotient[3] + EulerTotient[4] = 8 + 1 + 2 + 2 = 13

12. A similar approach as in problem number 6 could make a recursive relationship.

Problems 13 and 14 are research problems, purposely not well-defined, and so no particular solution can be written up as if it were "the right answer."

page 116, **Deep Arithmetic - Getting to One**

Which numbers give short sequences, and which give long sequences? What's the longest sequence you can find?
What if we said, for odd numbers, multiply by 3 and subtract 1? Does this always reach 1, too? How else could we change the rules?

Hint from the book:

To find short and long sequences, use a spreadsheet.
Check out 1 to 5 with the new rule.

Full solution:
I put the numbers 1 to 100 in column A. (A1 is 1, A2 is =A1+1, then copy down.) Then I put this formula in B1: =IF(INT(B1/2)=B1/2,B1/2,3*B1+1), and copied it across and down. I didn't see any obvious patterns. The number 97 takes 117 numbers to get down to 1.

With the changed rule (for odd numbers, multiply by 3 and subtract 1), try starting with 5: 5, 14, 7, 20, 10, 5, so we get a loop with only 5 numbers, which never goes back down to 1.

Starting with 1, we get 1, 2, 1.
Starting with 2 is the same loop.
Starting with 3, we get 3, 8, 4, 2, 1.
Starting with 4, we get 4, 2, 1.
Starting with 6, we get 6, 3, 8, 4, 2, 1.
Starting with 7, we get the same loop we got with 5.
Starting with 8, we get 8, 4, 2, 1.
Starting with 9, we get 9, 26, 13, 38, 19, 56, 28, 14, 7, 20, 10, 5, 14. So we've entered into the same loop we got with 5.
Starting with 10, we get the 5 loop.

Now our next question can become: Do all loops end up at 1 or at 5? Using a spreadsheet, we find a new loop for 17. Now I begin to wonder, are there an infinite number of loops?

Page 117, **Deep Arithmetic - Number Squares**

Will you always get to a row or square of all zeros? How many levels can you go before it settles down?
Can you find numbers that will take more than 7 rows to settle down to 0? Are there any sets of four numbers that will never settle down to 0?

Hint from the book:

> There are combinations that will make more than 7 rows. I was not able to find any myself, but knowing they're out there, maybe you can be more persistent.

Full solution:
Everything I tried gave me a seventh row with all zeros. So I checked online, and found a paper on this at math.utah.edu/mathcircle/notes/diffybox.pdf. You can get nine non-zero rows by starting with 0, 1, 1.5, 1.8 (or 0, 10, 15, 18). The number 1.8 is actually an approximation to the only real number solution to $x^3-x^2-x-1=0$, which plays a part in this problem, believe it or not. The paper is pretty readable - it was written up as notes for a math circle. Check it out. Using numbers closer to the solution to this equation gets longer sequences. 0, 1, 1.54, 1.83 gets 13 non-zero rows. The closer you get, the more rows you can get.

Will it always settle down? Yes, unless you use the solution to the equation above (call the solution q), in this sequence 0, 1, (q-1)q, q. Then you will get the same sequence of numbers back, and so it repeats forever. This comes from the paper cited above.

Page 140, **Self-Referential Puzzle**

Hint from the book:

> I think this is the hardest puzzle in the book. It seems impossible at first, but the bottom row is tangled together in ways we can untangle. How many times can a particular number appear in the puzzle? This limits the answers for many of the clues. Clue N is a good place to start.

Full solution:
Starting with clue N, the number of times 8 appears, we know that no number can appear more than four times (since no number can be repeated on a row or column), so the number here is 1, 2, 3, or 4 (not 0, since 0 isn't allowed). The number here is the same as the numbers in cells H and K. It cannot be 2, since clue E says there are exactly two 2's. It cannot be 3, because there would be these three 3's, and the only other possible place for a 3 is cell A, which can only have a 1 or a 2. But that would mean cell M would need to say 3 appears three times, which would make a fourth 3 – oops! Also, it can't be 4, because for 8 to appear four times, it would have to appear in the left column, but none of those cells can have an 8 (A is 1 or 2, E is 1, 2, or 3, I is no more than 6). We are left with 1 for this cell and cells H and K.

Now consider clue O. This number determines 3 things. Suppose cell P holds the 7. Then cell O is a 4, and the diagonals must each have four different numbers in them. The sum of the cells to the left of P would equal seven, which would make cell M have a 3 (M+1+4-1=7). So the sum of the bottom row would be 15, making cell A a 1. But now we would have two ones in a diagonal (at cells A and K), and couldn't have four different numbers in that diagonal. Therefore, cell P cannot hold the 7. We also know the 7 isn't in the first column. So cell O must be 2 or 3.

8	2	
×	|	
7	3	
5	6	**answer**

If cell M is a 4, then cell O cannot be a 3 (because that would put 7 in cell P). So cell M is 2 or 3, and cell O is 3 or 2, making cell P be 5, and cell L becomes 4. Now the bottom adds up to 11, so cell A is 1. 2, 3, and 5 are prime, so the bottom row has three primes, putting a 3 in cell D.

Cell E cannot be a 1, since cell A is. The 2's cannot be 3 apart vertically and diagonally, because they'd have to be in opposite corners, and both the top corners have other numbers (A is 1 and D is 3). So cell E must be a 2, and the other 2 must be two away diagonally (both vertically and horizontally), putting it in cell O, making cell M be the 3.

We know there's one 8, so cell B cannot be the 7. That puts the 7 in cell J. The only place left for the third 3 is cell G. Cell f must be a 1 or a 5 (since there are only two different digits on each diagonal), but 1 is taken (for its row and its column), so it must be a 5.

The parity of cell F is the same as the parity of its row (since the rest of the row sums to 12), and cell C also has the same parity. We know of ten odd numbers. If cell B has a 9, we'd have 11, plus possibly these two, but you can't take half of 11 or 13. So cell B is not a 9, and must be the 8 (which we decided appears once). Now we have 10 or 12 odd numbers, making half minus 1 equal 4 or 5. 4 is taken on row 3, so cell I must be 5. Cell C must be odd now, and cannot be 1 or 3 (they appear in its row), 7 (only appears once), or 9 (bigger than 8). It must be a 5. And we are done with what seemed impossible.

Page 148, **Vedic multiplication…**

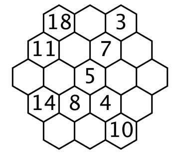
Suppose you need 8 x 7. 8 is 2 below 10, so 2 is the 'deficiency' for 8, and 7 is 3 below 10, so 3 is the deficiency for 7. Write your multiplication column style (with the 7 below the 8), and next to each number write its deficiency. Think of it like this:

You subtract crosswise 8-3, or 7 – 2, to get 5, the first digit of the answer. And you multiply vertically, 2 x 3, to get 6, the last digit of the answer. That's all you do. See how far the numbers are below 10, subtract one number's deficiency from the other number, and multiply the deficiencies together.

Hint from the book:

Use variable instead of 8 and 7. Describe the steps with your variables.

Full solution:



We want to multiply AxB. The deficiency for A is 10-A and the deficiency for B is 10-B. Subtracting crosswise gets us A-(10-B). Since this is the tens digit of the answer, let's represent that by multiplying this by 10: 10(A-10+B) = 10(A+B)-100. Multiply the deficiencies for the ones digit: (10-A)(10-B) = 100-10(A+B)+AB. Add these two parts together: 10(A+B)-100 + 100-10(A+B)+AB = AB. This shows us that the procedure will get the right answer. For what numbers is it useful? It's only going to be this easy when the deficiencies multiply to be less than ten.

Page 149, **To divide by 5…**

Hint from the book:

> What does moving the decimal do? What is the effect of doing both that and doubling?

Full solution:
This works because dividing by 5 is the same as dividing by 10 and then multiplying by 2, or multiplying by 2 and then dividing by 10. Double the number (number times 2), then move the decimal point to the left (divide by 10).

Page 153, **Magic Hexagon**

Fill in numbers so the grid has each number from 1 to 19 just once, with each row and diagonal adding to the same number. (Each cell is in one row and on two diagonals.)

Hint from the book:

> What number does each row and diagonal add up to? To find that we'll need to know the total for the sum of 1 to 19. (That has come up in *lots* of these puzzles!) You can make number pairs all the same, to find it: 1+19, 2+18… Then keep finding rows and diagonals that have just one open space.

Full solution:
There are 5 rows, and the numbers 1 to 19 add up to 190. (Remember how to do that from problem 1 on page 95?) So each row must add up to 38 (190/5).

Coming down diagonally to the right from the 18, we get a sum of 37, so we need a 1 in the open space. 18+3 = 21, so we need 17 in the open space in the top row. Now we can see that we need a 19 in the last spot in the second row. The fourth row gets a 12 at the end. The first diagonal coming down and left gets a 9 in its open spot. That allows us to put a 15 in the first spot on the bottom row, and then a 13 next to it. Now the diagonal down and right from the 11 gets a 6 in its second spot. The diagonal down and right from the 3 gets a 16 in its last spot. The last open spot now gets the last number left, a 2.

If you liked this puzzle, it might be a fun challenge to make your own. If you do, send it to me (mathanthologyeditor@gmail.com) and I'll post it on my blog.

Page 184, **Measuring with Paper**

Can you fold a standard 8 ½ by 11 inch sheet of paper to show an exact length of six inches - without using any measuring devices?

Hint from the book:

> Folding in half divides a measurement by two. What does folding on a diagonal do?

Full solution:
Sue's way: Half of 11 inches is 5.5 inches. If I fold the long side in half, and then fold diagonally, so that side is up against the top side, then the top is 3 inches longer. Fold back this extra 3 inches, then fold again to show 6 inches, which is double that.

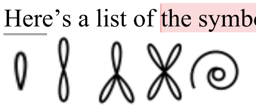
What ways did you find?

Page 195, **Alien Math**

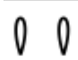
Hint from the book:

> These seem to be some sort of basic arithmetic problems. How many different digits do you see? It might help to make a list of them.

Full solution:
Here's a list of the symbols:



The first four look like petals to me. I see one, two, three, and four petals. The last one looks different. Maybe it could be zero. Then these problems would be in base five. Then



would be 1 five and 1 one, or six. If that's right, then the first problem would have the numbers 210 and 434, with 1144 below. Translating from base five to base ten, this

Sue VanHattum 8/9/13 12:18 PM
Comment [1]: These are png's. The original pdf is much crisper, but I didn't know how to get bits of it into my word doc.

Sue VanHattum 8/4/13 7:31 PM
Comment [2]: It's currently the second problem. We need to switch the first two.

would be (2*25+1*5 =) 55 and (4*25+3*5+4 =) 119, with (1*125+1*25+4*5+4 =) 174 below. That looks like an addition problem, so perhaps the hook symbol is the plus sign.

If we go with this translation scheme, then the next problem is 342 + 104 = 1001 (all in base five). Translating from base five to base ten would give us (3*25+4*5+2 =) 97 plus (1*25+4 =) 29 is (1*125=1 =) 126.

The fifth row uses a new symbol. Is it subtraction? If so, we'd have 413 – 320, in base five. That's 108 – 85 in base ten, which gives 23. 23 is written 43 in base five. Yep, that works.

On the next page we have a third symbol, perhaps it's multiplication? 32 x 41 would have partial products of 32 and (the 4 in 42 means 20, and 32 translates to 17 in base ten, 20x17= 340, which is written in base five as) 2330. Yep, it works. The final answer is 2412 (17+340 = 357, translate that to base five, it's 2*250+4*25+1*5+2). Yep, all good.

So, where the answers are in base 5,
3. 1002
4. 1001
5. 333
6. 242
7. 3003
8. 1000
9. 2200
10. 1140
11. 210
12. 10122

An upward facing hook represents subtraction.

15. 3302
16. 1141
17. 2232
18. 112

[Next page]
19. 313
20. 104
21. 412
22. 343
23. 321
24. 4143

A downward double hook represents multiplication.
In the layout, it is done in parts and then added.

e.g. in number 25, 32*41 is done as 32*1 + 5*(32*4) = 5*(233). i.e. 32 + 2330 = 2412.

27. 141
28. 402
29. 2002
30. 440
31. 220
32. 224
33. 42

Page 207, **Dartboard Problem (from Putting Myself In My Students' Shoes)**

You have a square dartboard. What is the probability that a randomly-thrown dart will land closer to the center of the dartboard than to an edge?

Hint from the book:

> The board has lots of symmetry. How can we take advantage of it? Closer means we're measuring distance. As soon as I start thinking about distances, I want to put in coordinates. (If you find a solution that doesn't use coordinates, please let me know!)

Full solution #1:
I want to find the edge of this space, which is where the distance to the center and an edge are equal. I want coordinates, so I'll put the center of the board at the origin, the right edge at x=1 and the top at y=1. I know that (1/2,0) is on the edge of my region, as is (0,1/2).

I wonder what point on the y=x line will be at the edge of my region?
Distance to the origin is $\sqrt{x^2+y^2}=\sqrt{x^2+x^2}=\sqrt{2}x$. Distance to the line x=1 is 1-x.
Setting these equal gives us $1-x=\sqrt{2}x$, or $(1-x)^2=2x^2$, which simplifies as follows:
$1-2x+x^2=2x^2 \Rightarrow x^2+2x-1=0 \Rightarrow x=\sqrt{2}-1$.

Hmm, that's just one point. Can I do something similar to describe all the points on the edge of this region? Distance to origin is $\sqrt{x^2+y^2}$. Distance to edge is 1-x (if 0<x<y).
Equating gives $\sqrt{x^2+y^2}=1-x \Rightarrow x^2+y^2=1-2x+x^2 \Rightarrow y^2=1-2x$.

Since I want area between two curves, the easiest way for me to do that is calculus. There's plenty of symmetry, and I think the other region in the first quadrant will be easier to integrate. So I'll look at the region where y>x, and then my equation of the edge will be $x^2=1-2y \Rightarrow 2y=1-x^2 \Rightarrow y=\dfrac{1-x^2}{2}$. Now I can integrate from x=0 to

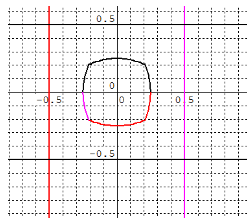
$x=\sqrt{2}-1$, using the area between this curve and y=x. (Note: The Greeks found the area under parabolas without calculus, so maybe this problem can be solved without calculus.)

I'll be finding the area of 1/8 of my region, so I need to multiply by 8. To find the probability, I need to divide my area by the whole area of the square, which is 4. In the end I'm multiply by 2. So $P=2\int_0^{\sqrt{2}-1}\left(\frac{1-x^2}{2}-x\right)dx=\int_0^{\sqrt{2}-1}\left(1-x^2-2x\right)dx=\left(x-\frac{x^3}{3}-x^2\right)\Bigg|_0^{\sqrt{2}-1}$

$=\left(\sqrt{2}-1\right)-\frac{1}{3}\left(\sqrt{2}-1\right)^3-\left(\sqrt{2}-1\right)^2=\sqrt{2}-1-\frac{1}{3}\left(3-2\sqrt{2}\right)\left(\sqrt{2}-1\right)-\left(3-2\sqrt{2}\right)$

$=\sqrt{2}-1-\frac{5\sqrt{2}-7}{3}-\left(3-2\sqrt{2}\right)=\sqrt{2}-1-\frac{5}{3}\sqrt{2}+\frac{7}{3}-3+2\sqrt{2}=\frac{4}{3}\sqrt{2}-\frac{5}{3}\approx .22$. So the probability of hitting closer to the center than to the edge (assuming your throws are no better than random) is about 22%.

Full solution #2:
First, you must describe the area that it covers. I thought of it like this: We want to know the radius of the loci, r, when the angle is t. Let us say t < 45 degrees from the horizontal. Then it must be nearest the right hand edge. So the distance between the right hand edge and it must be r. So then r cos(t) + r = w/2. Let us take w=1, as it otherwise doesn't matter. Then in that range, r = 1/(2(1+cos(t)).



By symmetry, and changing cosines and sines and such, the same thing works elsewhere. See picture.

Now to find the area of it, we need to integrate. If you have a polar equation, you can integrate the square of it then half it. Here's the solution on Wolfram alpha.

(We just use Mathematica, because doing it by hand is too much like hard work.) And so the probability is just 0.218951416497460....

Page 209, **What Number Am I?**

One of each pair is true, the other is false. What number is described?
1a. I have two digits
1b. I am even
2a. I contain the digit 7
2b. I am prime
3a. I am the product of two consecutive odd integers
3b. I am one more than a perfect square
4a. I am divisible by eleven
4b. I am one more than a perfect cube

5a. I am a perfect square
5b. I have three digits

Hint from the book:

> Experiment. Do any of the clues lead to a contradiction? If you check out the implications of clue 1a, you will get some useful information. Which other clues will this work with?

Full solution #1:

2-digit (1a) -> not 3-digit (not 5b) -> x^2 (5a) -> not prime (not 2b) -> contains 7 (2a), but no 2-digit perfect square contains a 7, so we know the number is even (1b), and does not have two digits.

The only even prime (2b) is the number 2, which is one more than a perfect square (3b), and one more than a perfect cube (4b), but is neither a perfect square (5a) nor a 3-digit number (5b). So the number cannot be prime, and must contain a 7 (2a).

It can't be one-digit (7 is prime and odd), so it must be at least 3-digit.

If it's one more than a perfect square, the number being squared would have to be odd (so the result is even). Trying 11 to 33, the only ones with a 7 are 170 and 730. 170 is neither divisible by eleven nor one more than a perfect cube. 730 is 9 cubed plus one. And it has 3 digits. 730 makes one statement of each pair false, and the other true.

Is there a better way to solve this?

Full solution #2:
Call the solution x.

Suppose [1a] holds. Then [5b] cannot hold and so [5a] must. But then neither [3a] nor [3b] can hold, as the product of two consecutive odd integers cannot possibly be a square.

Hence [1b] holds. It follows that either x=2 or x is not prime. x=2 fails, and so [2b] holds. [3a] cannot hold, as then x would be odd, so [3b] holds. [5a] cannot thus hold, so [5b] does.

Thus 170 and 730 are the only possibly candidates, by making a table (Is there a better way? I can't see one right now). It follows x=730, as neither [4a] nor [4b] holds for 170.

Page 213, **Octopus Logic**

Octopuses with an even number of arms always tell the truth; the ones with an odd number of arms always lie.

1. Four octopuses had a chat:

- *"I have eight arms," the green octopus bragged to the blue one. "You have only six!"*
- *"It is I who has eight arms," countered the blue octopus. "You have only seven!"*
- *"The blue one really has eight arms," the red octopus said, confirming the blue one's claim. He went on to boast, "I have nine arms!"*
- *"None of you have eight arms," interjected the striped octopus. "Only I have eight arms!"*

Who has exactly eight arms?

2. …only octopuses with six, seven, or eight arms are allowed to serve Neptune. Four of them were conversing:

- *The blue one said, "All together we have twenty-eight arms."*
- *The green one said, "All together we have twenty-seven arms."*
- *The yellow one said, "All together we have twenty-six arms."*
- *The red one said, "All together we have twenty-five arms."*

How many arms does each of them have?

3. The guards from the night shift at Neptune's palace started to argue:

- *The magenta one said, "All together we have thirty-one arms."*
- *The cyan one said, "No, we do not."*
- *The brown one said, "The beige one has six arms."*
- *The beige one said, "You, brown, are lying."*

Who is lying and who is telling the truth?

4. The last shift of guards at the palace has nothing better to do than count their arms:

- *The pink one said, "Gray and I have fifteen arms together."*
- *The gray one said, "Lavender and I have fourteen arms together."*
- *The lavender one said, "Turquoise and I have fourteen arms together."*
- *The turquoise one said, "Pink and I have fifteen arms together."*

How many arms does each one have?

Hint from the book:

Start with one hypothesis, and see if it leads to a contradiction.

Full solution:

1. Green says blue has six. If green is telling the truth then so is blue (even number of arms tells the truth). But blue disagrees with green. So green cannot be telling the truth. The red one cannot be telling the truth. If it had nine arms it would lie about it. So it doesn't have nine arms, and it is lying, so the blue one also does not have eight arms. Red doesn't since red is a liar. None of them has eight arms, so the first statement of the

striped octopus is true, which means its other statement is true. So the striped one has eight arms.

2. Since they all disagree on the total, only one can be telling the truth. Anyone who is lying has 7 arms. The one truth-teller could have 6 or 8. The total must be 27 or 29. Twenty nine wasn't mentioned. So the green one is telling the truth and has 6 arms, with the others having 7.

3. If brown is telling the truth, then beige will tell the truth and they won't disagree. Since they do disagree, brown is lying and that means beige is telling the truth. Since brown is lying, beige must have 8 arms. Brown has 7. For magenta's claim to be true, both magenta and cyan need 8 arms, but then cyan would tell the truth and agree. So magenta lies, and has 7 arms. Cyan tells the truth and has 6 or 8 arms.

4. If Pink is telling the truth, then pink must have 8 arms and Gray must have 7, which would make Gray a liar, so Gray and Lavender together couldn't have 14, which would mean Lavender would be a truth teller. So Turquoise would also have an even number of arms (to add up to 14) and be a truth teller too. But then Turquoise and Pink could not have an odd number of arms. We have come to a contradiction, so we know that Pink is lying and has 7 arms. We also know that Gray cannot have eight arms (which would make 15 between them).

If Gray is telling the truth, it must have 6 arms, which would give Lavender 8 arms, and that would give Turquoise 6 arms, but that would give Turquoise plus Pink 13 arms, and we have another contradiction. So we know Gray is a liar and has 7 arms. We also know that Lavender does not have 7 arms, and must tell the truth. If Lavender has 8 arms, then Turquoise has 6. If Lavender has 6 arms, then Turquoise has 8. Either way, Turquoise tells the truth. If Turquoise and Pink have 15 arms together, then Turquoise must have eight, giving Lavender 6.

Page 217, **Coloring Cubes**

Take a cube whose edges are n units long, paint its surface completely, cut it into unit cubes whose edges are 1 unit long, and then ask how many unit cubes are painted in each way.

1. *If you start with a two-unit cube, how many of the resulting unit cubes are completely unpainted? Painted on just one side? Two sides? Three sides?*
2. *Starting with a three-unit cube, answer the same questions.*
3. *Repeat for a four-unit cube.*
4. *Repeat for an n unit cube. Can you find the pattern?*

Now we're going to first cut the cube into unit cubes, paint them, and then put them back together. But there's a little catch! We want to paint with several different colors, in fact as many as possible, so that the pieces can be reassembled to make a cube whose outside is all one color.

5. *You can cut a two-unit cube into eight one-unit cubes, and paint them with two colors in such a way that you could put them back together into a two-unit cube of either color. How should you paint them?*
6. *With a three-unit cube, cut into twenty-seven one-unit cubes, can you paint them with three colors in such a way that they can be put back together into a three-unit cube of any of the three colors? If so, how? If not, why not?*
7. *Does this generalize? Starting with an n-unit cube, can you cut it into one-unit cubes and paint them with n colors in such a way that they can be reassembled into an n-unit cube of any one of the colors?*

Hint from the book:

Where are the ones painted on three sides? …two sides? …one side? … no sides? Does that help you figure out how many?

Full solution:
1. All of the eight cubes are painted on all three sides.
2. Eight are painted on all three sides. Twelve are painted on two sides. Six are painted on one side. The one in the center is unpainted.
3. Eight are painted on all three sides. Twenty four are painted on two sides. Twenty four are painted on one side. Eight are unpainted.
4. The eight corner cubes are painted on all three sides. 12(n-2) are painted on two sides. $6(n-2)^2$ are painted on one side. And $(n-2)^3$ are unpainted. (Adding these up to double check does get a sum of n^3.)
5. Paint the three faces that touch at one vertex all of one color, and the other three faces all of the other color.
6. Yes, it's possible. Suppose the colors are red, green, and blue (R,G,B). We need eight cubes that have three sides painted, for each of the three colors. We also need twelve painted on two sides, for each of the three colors. And six painted on one side, for each of three colors. We can paint cubes three ways: three of one color, three of another, as we did on the two-unit cube, two of each of the three colors (front and top, left and bottom, back and right), or 3 of one color, 2 of another, 1 of the third. We need 18 total of just one side, so we'll do the 3,2,1 pattern on 18 of the cubes. We need six more of the three-sides, so we'll do three of 3,3. Then we have six left to do 2,2,2.
7. We have $6*n^3$ total faces, and we need to paint $6*n^2$ outer faces, in each of n colors. We need:
n*8 corner pieces, painted on three sides (24n faces)
n*12(n-2) painted on two sides (24n(n-2) faces)
$n*6(n-2)^2$ painted on one side ($6n(n-2)^2$ faces)
Since the one-siders can't all be on different cubes now (we would need more cubes than we have), we must add another painting scheme into the mix, making some cubes have more than one color for one side each. For 4 colors, we can do 24 of the cubes with one side R, two sides G, one side, B, and two sides Y (1,2,1,2), and 24 more with the ratios reversed. Then do 16 with three of one color, three of another.
For the n-unit cube, we'd have 6n(n-2)(n-4) more of the one-siders than the two-siders, so we paint n(n-2)(n-4) of them in the 1,1,1,1,1,1 pattern. Then we'd paint 6n(n-2) of them in the 2,1,2,1 pattern, and 4n of them in the 3,3 pattern.

Page 228, **A Little Math Magic**

Think of a number,
Now square your number.
Take your whole result, and throw it out, except for the rightmost digit, the units digit.
You should have a one digit number.
Now take that one digit number, and square it once again,
And again throw it all out, except the rightmost digit.
Now, take that last one digit number, and multiply it by your original number

Hint from book:

If you want to *understand* Jonathan's trick, there are three steps:

- figure out the trick (try using a spreadsheet to figure out the results and then to work backward),
- prove Fermat's Little Theorem (a good explanation is at youtube.com/watch?v=w0ZQvZLx2KA), and
- then figure out how the theorem predicts the trick (which uses the fact that $a^4 \equiv 1 \pmod 5$ for all numbers except multiples of 5).

Full solution:
Step one: Set up a spreadsheet that does each step that Jonathan told his students to do, for all the numbers from 1 through 30:
Cell A1: 1
Cell A2: =A1+1
Cell B1: =A1^2
Cell C1: =B1-10*INT(B1/10)
Cell D1: =C1^2
Cell E1: =D1-10*INT(D1/10)
Cell F1: =A1*E1
Copy cells B1 to F1 downward, to row 2.
Now copy cells A2 to F2 downward, to row 30.

You can see that odd numbers end up with the original number, except that multiples of five end up with five times the original number, and even numbers end up with six times the original number, except that multiples of ten all end up at zero. So the first time someone says zero, your best guess may be ten. After that, guess any multiple of ten. If the student gives you an odd number, you know that's their original, except if they give you a multiple of five you divide by five to get the original. If they give you an even number, divide by six.

Step two: Watch the video at youtube.com/watch?v=w0ZQvZLx2KA. Try to write each step down while explaining it to yourself. If you get stuck, and want to understand it better, feel free to email Sue at mathanthologyeditor@gmail.com.

Step three: Since 10 = 2*5, x^4 mod 10 …

Page 238, **Personal Ad for 21**

The Numberland News runs personal ads. 21 was looking for a new friend and put an ad in: Two-digit, semi-prime, triangular, Fibonacci number seeks same. I'm a binary palindrome, what about you?
Will 21 find a friend?

Hint from the book:

> Fibonacci numbers may be the easiest to find. List the two-digit ones, and then see which fit the other criteria.

Full solution:
The two digit Fibonacci numbers are 13, 21, 34, 55, 89.
Of which 21, 34, 55 are semiprimes. (Semiprimes are two primes multiplied together. 55 = 5*11, 34 = 2*17, 21 = 3*7. 13 and 89 are in fact primes.)
Of which only 21 and 55 are triangular.
21 in binary is 10101, and 55 in binary is 110111. So 55 isn't a binary palindrome. I think 21 and 55 will be good friends, since 55 *is* a decimal palindrome.